

SOME IDENTITIES OF PARTIALLY DEGENERATE TOUCHARD POLYNOMIALS ARISING FROM DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider the partially degenerate Touchard polynomials which are derived from the generating function. In addition, we present differential equations for the generating function of the partially degenerate Touchard polynomials and give some new identities of these polynomials arising from the differential equations.

1. Introduction

It is well known that the Touchard polynomials (also called exponential or Bell polynomials) $T_n(x)$ play an important role in statistics and probability. They are given by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \quad (\text{see [2 – 15]}). \quad (1.1)$$

From (1.1), we note that

$$\begin{aligned} e^{x(e^t-1)} &= \sum_{m=0}^{\infty} \frac{x^m}{m!} (e^t - 1)^m = \sum_{m=0}^{\infty} x^m \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n x^m S_2(n, m) \right) \frac{t^n}{n!}, \end{aligned} \quad (1.2)$$

where $S_2(n, m)$ is the Stirling number of the second kind. Thus, by (1.1) and (1.2), we get

$$T_n(x) = \sum_{m=0}^n S_2(n, m) x^m, \quad (n \geq 0), \quad (\text{see [12, 13, 14]}). \quad (1.3)$$

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When $x = 1$, $T_n = T_n(1)$ are called Touchard numbers. From (1.1), we have

$$T_n(x) = e^{-x} \sum_{k=0}^{\infty} k^n \frac{x^k}{k!} = \sum_{k=0}^n S_2(n, k) x^k, \quad (n \geq 0) \quad (1.4)$$

and

$$T_n(x+y) = \sum_{l=0}^n \binom{n}{l} T_l(x) T_{n-l}(y), \quad (\text{see [13, 14, 15, 16]}). \quad (1.5)$$

It is not difficult to show that

$$\begin{aligned} T_0(x) &= 1, \quad T_1(x) = x, \quad T_2(x) = x^2 + x, \quad T_3(x) = x^3 + 3x^2 + x, \\ T_4(x) &= x^4 + 6x^3 + 7x^2 + x, \quad T_5(x) = x^5 + 10x^4 + 25x^3 + 15x^2 + x, \dots \end{aligned}$$

For $\lambda \in \mathbb{R}$, L. Carlitz introduced the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see [1, 16]}). \quad (1.6)$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers (see [1, 16]). Note that $\lim_{\lambda \rightarrow 0} \beta_{n,\lambda}(x) = B_n(x)$, ($n \geq 0$), are the Bernoulli polynomials given by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [16]}). \quad (1.7)$$

With the viewpoint of Carlitz, Kim introduced the degenerate Bell polynomials (also called degenerate Touchard polynomials) which are given by the generating function

$$(1+\lambda)^{\frac{x}{\lambda}} \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 \right) = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x) \frac{t^n}{n!}, \quad (\text{see [16]}). \quad (1.8)$$

Note that $\lim_{\lambda \rightarrow 0} Bel_{n,\lambda}(x) = T_n(x)$, ($n \geq 0$). When $x = 1$, $Bel_{n,\lambda}(1) = Bel_{n,\lambda}$ are called the degenerate Bell numbers. In this paper, we consider the partially degenerate Touchard polynomials which are derived from the generating function. In addition, we present differential equations for the generating function of the partially degenerate Touchard polynomials and give some new identities of these polynomials arising from the differential equations.

2. Some identities of partially degenerate Touchard polynomials arising from differential equations

Let us consider the partially degenerate Touchard polynomials which are given by the generating function

$$F = F(t, x|\lambda) = e^{x((1+\lambda t)^{\frac{1}{\lambda}} - 1)} = \sum_{n=0}^{\infty} T_{n,\lambda}(x) \frac{t^n}{n!}. \tag{2.1}$$

When $x = 1$, $T_{n,\lambda} = T_{n,\lambda}(1)$ are called the partially degenerate Touchard numbers. From (2.1), we note that

$$\begin{aligned} T_{n,\lambda}(x) &= \sum_{k=0}^n \sum_{m=0}^k S_2(k, m) S_1(n, k) \lambda^{n-k} x^m \\ &= e^{-x} \sum_{k=0}^{\infty} \frac{1}{k!} (k|\lambda)_n x^k, \quad (n \geq 0), \end{aligned} \tag{2.2}$$

where $(x|\lambda)_0 = 1$, $(x|\lambda)_n = x(x - \lambda)(x - 2\lambda) \cdots (x - (n - 1)\lambda)$, $(n \geq 1)$, and $S_1(n, k)$ is the stirling number of the first kind. From (2.1), we have

$$\begin{aligned} F^{(1)} &= \frac{d}{dt} F(t, x|\lambda) = \frac{x}{\lambda} \frac{\lambda}{1 + \lambda t} e^{x((1+\lambda t)^{\frac{1}{\lambda}} - 1)} \cdot (1 + \lambda t)^{\frac{1}{\lambda}} \\ &= x(1 + \lambda t)^{\frac{1}{\lambda} - 1} F, \end{aligned} \tag{2.3}$$

$$\begin{aligned} F^{(2)} &= \frac{d}{dt} F^{(1)} = x \left(\frac{1 - \lambda}{\lambda} \right) \frac{\lambda}{1 + \lambda t} (1 + \lambda t)^{\frac{1}{\lambda} - 1} F + x(1 + \lambda t)^{\frac{1}{\lambda} - 1} F^{(1)} \\ &= x(1 - \lambda)(1 + \lambda t)^{\frac{1}{\lambda} - 2} F + x(1 + \lambda t)^{\frac{1}{\lambda} - 1} \left(x(1 + \lambda t)^{\frac{1}{\lambda} - 1} F \right) \\ &= \left(x(1 - \lambda)(1 + \lambda t)^{\frac{1}{\lambda} - 2} + x^2(1 + \lambda t)^{\frac{2}{\lambda} - 2} \right) F \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} F^{(3)} &= \frac{d}{dt} F^{(2)} \\ &= \left(x(1 - \lambda)(1 - 2\lambda)(1 + \lambda t)^{\frac{1}{\lambda} - 3} + 3x^2(1 - \lambda)(1 + \lambda t)^{\frac{2}{\lambda} - 3} + x^3(1 + \lambda t)^{\frac{3}{\lambda} - 3} \right) F. \end{aligned} \tag{2.5}$$

Continuing this process, we can set

$$F^{(N)} = \left(\frac{d}{dt} \right)^N F(t, x|\lambda) = \left(\sum_{i=1}^N b_i(N, \lambda) x^i (1 + \lambda t)^{\frac{i}{\lambda} - N} \right) F, \tag{2.6}$$

where $N = 1, 2, 3, \dots$. To determine the coefficients $b_i(N, \lambda)$ in (2.6), we take the derivative with respect to t on both sides of (2.6) as follows:

$$\begin{aligned}
 F^{(N+1)} &= \left(\frac{d}{dt}\right) F^{(N)} \\
 &= \sum_{i=1}^N b_i(N, \lambda) x^i \left(\frac{i - N\lambda}{\lambda}\right) \lambda (1 + \lambda t)^{\frac{i}{\lambda} - N - 1} F + \sum_{i=1}^N b_i(N, \lambda) x^i (1 + \lambda t)^{\frac{i}{\lambda} - N} F^{(1)} \\
 &= \sum_{i=1}^N b_i(N, \lambda) x^i (i - N\lambda) (1 + \lambda t)^{\frac{i}{\lambda} - N - 1} F + \sum_{i=1}^N b_i(N, \lambda) x^{i+1} (1 + \lambda t)^{\frac{i+1}{\lambda} - N - 1} F \\
 &= \sum_{i=1}^N b_i(N, \lambda) x^i (i - N\lambda) (1 + \lambda t)^{\frac{i}{\lambda} - N - 1} F + \sum_{i=2}^{N+1} b_{i-1}(N, \lambda) x^i (1 + \lambda t)^{\frac{i}{\lambda} - N - 1} F.
 \end{aligned} \tag{2.7}$$

By replacing N by $N + 1$ in (2.6), we get

$$F^{(N+1)} = \sum_{i=1}^{N+1} b_i(N+1, \lambda) x^i (1 + \lambda t)^{\frac{i}{\lambda} - N - 1} F. \tag{2.8}$$

Comparing the coefficients on both sides of (2.7) and (2.8), we have

$$b_1(N+1, \lambda) = (1 - N\lambda)b_1(N, \lambda), \quad b_{N+1}(N+1, \lambda) = b_N(N, \lambda) \tag{2.9}$$

and

$$b_i(N+1, \lambda) = b_{i-1}(N, \lambda) + (i - N\lambda)b_i(N, \lambda), \tag{2.10}$$

where $2 \leq i \leq N$. From (2.3), we have

$$x(1 + \lambda t)^{\frac{1}{\lambda} - 1} F = F^{(1)} = b_1(1, \lambda) x(1 + \lambda t)^{\frac{1}{\lambda} - 1} F \tag{2.11}$$

Thus, by (2.11), we get

$$b_1(1, \lambda) = 1. \tag{2.12}$$

By (2.4), we easily get

$$\begin{aligned}
 &\left(x(1 - \lambda)(1 + \lambda t)^{\frac{1}{\lambda} - 2} + x^2(1 + \lambda t)^{\frac{2}{\lambda} - 2}\right) F = F^{(2)} \\
 &= \left(b_1(2, \lambda) x(1 + \lambda t)^{\frac{1}{\lambda} - 2} + b_2(2, \lambda) x^2(1 + \lambda t)^{\frac{2}{\lambda} - 2}\right) F.
 \end{aligned} \tag{2.13}$$

From (2.13), we have

$$b_1(2, \lambda) = (1 - \lambda), \quad b_2(2, \lambda) = 1. \tag{2.14}$$

It is easy to show that

$$b_{N+1}(N+1, \lambda) = b_N(N, \lambda) = b_{N-1}(N-1, \lambda) = \dots = b_1(1, \lambda) = 1, \tag{2.15}$$

and

$$\begin{aligned}
 b_1(N+1, \lambda) &= (1 - N\lambda)b_1(N, \lambda) = (1 - N\lambda)(1 - (N-1)\lambda)b_1(N-1, \lambda) = \dots \\
 &= (1 - N\lambda)(1 - (N-1)\lambda) \dots (1 - \lambda)b_1(1, \lambda) \\
 &= \langle 1 - N\lambda | \lambda \rangle_N,
 \end{aligned}
 \tag{2.16}$$

where

$$\langle x | \alpha \rangle_0 = 1, \quad \langle x | \alpha \rangle_N = x(x + \alpha) \dots (x + (N-1)\alpha), \quad (N \geq 1).$$

Now, we give an explicit expression for $b_i(N+1, \lambda)$, where $2 \leq i \leq N$. For $i = 2, 3, 4$ in (2.10), we have

$$\begin{aligned}
 b_2(N+1, \lambda) &= (2 - N\lambda)b_2(N, \lambda) + b_1(N, \lambda) \\
 &= (2 - N\lambda)(2 - (N-1)\lambda)b_2(N-1, \lambda) + (2 - N\lambda)b_1(N-1, \lambda) + b_1(N, \lambda) \\
 &= \dots \\
 &= \sum_{k_1=0}^{N-1} \langle 2 - N\lambda | \lambda \rangle_{k_1} b_1(N - k_1, \lambda) \\
 &= \sum_{k_1=0}^{N-1} \langle 2 - N\lambda | \lambda \rangle_{k_1} \langle 1 - (N - k_1 - 1)\lambda | \lambda \rangle_{N - k_1 - 1},
 \end{aligned}
 \tag{2.17}$$

$$\begin{aligned}
 b_3(N+1, \lambda) &= (3 - N\lambda)b_3(N, \lambda) + b_2(N, \lambda) \\
 &= (3 - N\lambda)(3 - (N-1)\lambda)b_3(N-1, \lambda) + (3 - N\lambda)b_2(N-1, \lambda) + b_2(N, \lambda) \\
 &= \dots \\
 &= \sum_{k_2=0}^{N-2} \langle 3 - N\lambda | \lambda \rangle_{k_2} b_2(N - k_2, \lambda) \\
 &= \sum_{k_2=0}^{N-2} \sum_{k_1=0}^{N-2-k_2} \langle 3 - N\lambda | \lambda \rangle_{k_2} \langle 2 - (N - k_2 - 1)\lambda | \lambda \rangle_{k_1} \\
 &\quad \times \langle 1 - (N - k_2 - k_1 - 2)\lambda | \lambda \rangle_{N - k_2 - k_1 - 2}
 \end{aligned}
 \tag{2.18}$$

and

$$\begin{aligned}
 b_4(N+1, \lambda) &= b_3(N, \lambda) + (4 - N\lambda)b_4(N, \lambda) \\
 &= (4 - N\lambda)(4 - (N-1)\lambda)b_4(N-1, \lambda) + (4 - N\lambda)b_3(N-1, \lambda) + b_3(N, \lambda) \\
 &= \dots \\
 &= \sum_{k_3=0}^{N-3} \langle 4 - N\lambda | \lambda \rangle_{k_3} b_3(N - k_3, \lambda) \\
 &= \sum_{k_3=0}^{N-3} \sum_{k_2=0}^{N-3-k_3} \sum_{k_1=0}^{N-3-k_3-k_2} \langle 4 - N\lambda | \lambda \rangle_{k_3} \langle 3 - (N - k_3 - 1)\lambda | \lambda \rangle_{k_2} \\
 &\quad \times \langle 2 - (N - k_3 - k_2 - 2)\lambda | \lambda \rangle_{k_1} \langle 1 - (N - k_3 - k_2 - k_1 - 3)\lambda | \lambda \rangle_{N-k_3-k_2-k_1-3}.
 \end{aligned} \tag{2.19}$$

Continuing this process, we have

$$\begin{aligned}
 b_i(N+1, \lambda) &= \sum_{k_{i-1}=0}^{N-i+1} \sum_{k_{i-2}=0}^{N-i+1-k_{i-1}} \dots \sum_{k_1=0}^{N-i+1-k_{i-1}-\dots-k_2} \left(\prod_{l=2}^i \langle l - (N - \sum_{j=l}^{i-1} k_j - i + l)\lambda | \lambda \rangle_{k_{l-1}} \right) \\
 &\quad \times \left(\langle 1 - (N - \sum_{j=1}^{i-1} k_j - i + 1)\lambda | \lambda \rangle_{N - \sum_{j=1}^{i-1} k_j - i + 1} \right).
 \end{aligned} \tag{2.20}$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $N \in \mathbb{N}$, the differential equations

$$F^{(N)} = \left(\sum_{i=1}^N b_i(N, \lambda) x^i (1 + \lambda t)^{\frac{i}{\lambda} - N} \right) F$$

have the solution

$$F = F(t, x | \lambda) = e^{x((1+\lambda t)^{\frac{1}{\lambda}} - 1)},$$

where $b_1(N, \lambda) = \langle 1 - (N-1)\lambda | \lambda \rangle_{N-1}$,

$$\begin{aligned}
 b_i(N, \lambda) &= \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-i-k_{i-1}} \dots \sum_{k_1=0}^{N-i-k_{i-1}-\dots-k_2} \left(\prod_{l=2}^i \langle l - (N - \sum_{j=l}^{i-1} k_j - i - 1 + l)\lambda | \lambda \rangle_{k_{l-1}} \right) \\
 &\quad \times \left(\langle 1 - (N - \sum_{j=1}^{i-1} k_j - i)\lambda | \lambda \rangle_{N - \sum_{j=1}^{i-1} k_j - i} \right), \quad (2 \leq i \leq N).
 \end{aligned}$$

Recall that the partially degenerate Touchard polynomials $T_{n,\lambda}(x)$ are defined by the generating function

$$F = F(t, x|\lambda) = e^{x((1+\lambda t)^{\frac{1}{\lambda}} - 1)} = \sum_{n=0}^{\infty} T_{n,\lambda}(x) \frac{t^n}{n!}. \tag{2.21}$$

Thus, by (2.21), we easily get

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t, x|\lambda) = \sum_{n=0}^{\infty} T_{n+N,\lambda}(x) \frac{t^n}{n!}, \quad (N \in \mathbb{N}). \tag{2.22}$$

From Theorem 1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n+N,\lambda}(x) \frac{t^n}{n!} &= F^{(N)} = \left(\frac{d}{dt}\right)^N F(t, x|\lambda) \\ &= \left(\sum_{i=1}^N b_i(N, \lambda) x^i (1 + \lambda t)^{\frac{i}{\lambda} - N}\right) F \\ &= \sum_{i=1}^N b_i(N, \lambda) x^i \left(\sum_{l=0}^{\infty} \binom{i}{\lambda} \lambda^l \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} T_{m,\lambda}(x) \frac{t^m}{m!}\right) \\ &= \sum_{i=1}^N b_i(N, \lambda) x^i \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} \left(\frac{i}{\lambda} - N\right)_l \lambda^l T_{n-l,\lambda}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^N \sum_{l=0}^n \binom{n}{l} \left(\frac{i}{\lambda} - N\right)_l \lambda^l b_i(N, \lambda) x^i T_{n-l,\lambda}(x) \right\} \frac{t^n}{n!} \end{aligned} \tag{2.23}$$

Therefore, by comparing the coefficients on both sides of (2.23), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$ and $N \in \mathbb{N}$, we have

$$\begin{aligned} T_{n+N,\lambda}(x) &= \sum_{i=2}^N \sum_{l=0}^n \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-i-k_{i-1}} \cdots \sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} \binom{n}{l} \left(\frac{i}{\lambda} - N\right)_l \lambda^l x^i T_{n-l,\lambda}(x) \\ &\times \left(\prod_{l=2}^i \langle l - (N - \sum_{j=l}^{i-1} k_j - i - 1 + l)\lambda | \lambda \rangle_{k_{l-1}} \right) \left(\langle 1 - (N - \sum_{j=1}^{i-1} k_j - i)\lambda | \lambda \rangle_{N - \sum_{j=1}^{i-1} k_j - i} \right) \\ &+ \sum_{l=0}^n \binom{n}{l} \left(\frac{1}{\lambda} - N\right)_l \lambda^l \langle 1 - (N - 1)\lambda | \lambda \rangle_{N-1} x T_{n-l,\lambda}(x). \end{aligned}$$

Remark. Note that

$$\begin{aligned}
T_{n+N}(x) &= \lim_{\lambda \rightarrow 0} T_{n+N,\lambda}(x) \\
&= \sum_{i=2}^N \sum_{l=0}^n \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-i-k_{i-1}} \cdots \sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} \binom{n}{l} i^l x^i T_{n-l}(x) 2^{k_1} 3^{k_2} \cdots i^{k_{i-1}} \\
&\quad + x \sum_{l=0}^n \binom{n}{l} T_{n-l}.
\end{aligned}$$

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